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Note

A Relation between Peano Kernels and a Theorem of Bernstein*

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1. INTRODUCTION

First we recall a definition which we proposed in an earlier paper [4].

DEFINITION 1. A semi-norm E_n on $C^{n+1}[a, b]$ is said to satisfy Property B of order n if

$$|f^{(n+1)}(x)| \leqslant g^{(n+1)}(x), \qquad a \leqslant x \leqslant b, \tag{1.1}$$

implies that $E_n(f) \leqslant E_n(g)$.

The term "Property B" is used to mark its connection with S. N. Bernstein [1] in the special case of the semi-norm

$$E_n(f) = \inf_{q \in P_n} \|f - q\|_{\infty}. \tag{1.2}$$

We also [4] proved the following generalization of a theorem of Bernstein.

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THEOREM 1. *If E_n is a semi-norm which satisfies Property B of order n, then, for each $f \in C^{n+1}[a, b]$, there exists $\xi \in (a, b)$ such that*

$$E_n(f) = \frac{1}{(n+1)!} |f^{(n+1)}(\xi)| \cdot E_n(x^{n+1}). \tag{1.3}$$

Bernstein's special case of Theorem 1 refers to the semi-norm defined by (1.2). See, for example, Meinardus [6].

We now recall the definition of a *monotone* norm (cf. Cheney [2, p. 40]).

DEFINITION 2. A norm $\|\cdot\|$ defined on $C[a, b]$ is said to be monotone if $|f(x)| \leqslant |g(x)|$, $a \leqslant x \leqslant b$, implies that $\|f\| \leqslant \|g\|$.

Kimchi and Richter-Dyn [5] have shown that if

$$E_n(f) = \inf_{q \in P_n} \|f - q\| \tag{1.4}$$

and $\|\cdot\|$ is a monotone norm, then the semi-norm E_n satisfies Property B. In view of Theorem 1, this implies that (1.3) holds for all semi-norms of the form (1.4) where $\|\cdot\|$ is a monotone norm. This generalizes an earlier result of Phillips [7], who showed that (1.3) holds for all semi-norms of the form (1.4) where $\|\cdot\|$ is any of the p-norms, $1 \leqslant p \leqslant \infty$.

In the present note, we establish a connection between Property B, expressions of the type (1.3) and Peano kernels.

2. PROPERTY B AND PEANO KERNELS

We begin by stating the Peano Kernel Theorem: If L_n is a Peano functional on $C^{n+1}[a, b]$ vanishing on P_n then, for any $f \in C^{n+1}[a, b]$,

$$L_n(f) = \int_a^b f^{(n+1)}(t)\, K(t)\, dt \tag{2.1}$$

where K is the Peano kernel

$$K(t) = \frac{1}{n!} L_n((x - t)_+^n).$$

See, for example, Davis [3, pp. 69–71]. This theorem has a well-known Corollary: If the kernel K does not change sign on $[a, b]$, there exists some $\xi \in (a, b)$ such that

$$L_n(f) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)\, L_n(x^{n+1}). \tag{2.2}$$

Although the corollary is generally quoted in the literature in the above form, the condition stated in the corollary is, in fact, both necessary and sufficient. We therefore restate it as follows.

PEANO COROLLARY. *If L_n is a Peano functional on $C^{n+1}[a,b]$ vanishing on P_n, then there exists some $\xi \in (a,b)$ such that (2.2) holds if and only if the kernel K does not change sign on $[a,b]$ (except possibly on a subset of measure zero).*

Proof. The proof of the "if" part is well known. To prove the "only if" part let us suppose that the kernel K does change sign and that

$$\int_a^b K(t)\,dt \geqslant 0.$$

It then follows from (2.1) that $L_n(x^{n+1}) \geqslant 0$. Now choose a function f such that $f^{(n+1)}(x) > 0$ on $[a,b]$ and $L_n(f) < 0$. This is clearly possible, as may be seen from (2.1), since $K(t) < 0$ on some non-trivial subinterval of $[a,b]$. In such a case there is no ξ such that (2.2) holds and this completes the proof.

Now, given any Peano functional L_n on $C^{n+1}[a,b]$ vanishing on P_n, we can define a semi-norm

$$E_n(f) = |L_n(f)|$$

and so obtain from (2.2) a relation of the form (1.3). In order to pursue this connection, we require one further definition.

DEFINITION 3. A semi-norm E_n on $C^{n+1}[a,b]$ is said to have a kernel if there exists an integrable function K on $[a,b]$ such that

$$E_n(f) = \left|\int_a^b f^{(n+1)}(t)\,K(t)\,dt\right|, \qquad \forall f \in C^{n+1}[a,b],$$

where K is independent of f.

Thus for semi-norms as defined by $E_n(f) = |L_n(f)|$, where L_n is a Peano functional, the kernel is simply the usual Peano kernel.

We conclude with a result which reveals a link between kernels of semi-norms and Property B.

THEOREM 2. *If a semi-norm E_n on $C^{n+1}[a,b]$ has a kernel K, then K is of constant sign if and only if E_n satisfies Property B of order n.*

Proof. Suppose K is of constant sign and f, g satisfy (1.1). Then

$$E_n(f) = \left| \int_a^b f^{(n+1)}(t)\, K(t)\, dt \right|$$
$$\leqslant \int_a^b |f^{(n+1)}(t)| \cdot |K(t)|\, dt$$
$$\leqslant \int_a^b g^{(n+1)}(t) \cdot |K(t)|\, dt$$
$$= \left| \int_a^b g^{(n+1)}(t)\, K(t)\, dt \right| = E_n(g).$$

Hence E_n satisfies Property B of order n. The converse follows from Theorem 1 and an argument similar to that used in the "only if" portion of the proof of the Peano Corollary.

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